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Technical Report No. 3

GEOMETRIC DISCONTINUITIES IN ELASTOSTATICS

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M. K. Kassir[2] and G. C. Sih[3]

Abstract. Three-dimensional elastostatic problems for an infinite solid with geometric discontinuities are formulated and solved with the aid of potential functions. In the problem of a linearly varying pressure specified over a plane region bounded by an ellipse, use is made of the gravitational potential at an exterior point of a homogeneous elliptical disk. The problem of prescribing displacements on the surfaces of disconti-nuity is governed by the Newtonian potential of a simple layer distributed over a disk in the shape of the region of discontinuity. The mass density of the disk is proportional to the prescribed normal displacement. For an "elliptically-shaped" region, the application of the symmetrical form of ellipsoidal coordinates leads to an integral equation of the Abel type for the potential function. It is shown that if the displacements normal to the elliptical plane are given by $(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})^{1/2} Q_n(x^2,y^2)$, where $Q_n(x^2,y^2)$ is a polynomial of degree n in x^2 and y^2, then the corresponding normal stress acting over the ellipse is also a polynomial, $P_n(x^2,y^2)$, of the same degree in x^2 and y^2. In the case of a circular region of discontinuity, the solution can be carried out for any arbitrary value of n. The results are useful in the prediction of the stability behavior of elastic solids containing geometric discontinuities.

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1. Introduction. The redistribution of stresses due to the presence of geometric discontinuities in elastic solids has been the subject of many past discussions. Among the three-dimensional problems considered previously, the majority of them pertains to geometries and applied loadings that possess axial symmetry. A collection of previous work can be found in the notes of Sneddon [1] and the references therein. The non-symmetric problems involving geometric discontinuities are more difficult and have been investigated only in a limited number of special cases.

A knowledge of the classical potential theory is pertinent to the formulation of boundary value problems with planes of discontinuities in the three-dimensional space. For definiteness sake, the region of discontinuity will be assumed to take the form of a plane ellipse described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , \quad z = 0, \tag{1.1}$$

which is referred to a set of cartesian coordinates x, y, and z. The major and minor semi-axes of the ellipse are denoted by a and b, respectively. If pressures, $p(x,y)$, are applied symmetrically to the upper and lower sides of the elliptical plane, the problem reduces to the determination of a single harmonic function [2] , $f(x,y,z)$, satisfying the boundary conditions

$$\frac{\partial^2 f}{\partial z^2} = -\frac{p(x,y)}{2\mu} \quad , \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \, , \; z = 0\right) \tag{1.2}$$

$$\frac{\partial f}{\partial z} = 0 \, , \qquad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1 \, , \; z = 0\right) \tag{1.3}$$

where μ is the shear modulus of the elastic solid. Since $f(x,y,z)$ is harmonic, these conditions may be expressed in the equivalent form

$$f = p^*(x,y) \quad , \qquad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \, , \; z = 0\right) \tag{1.4}$$

$$\frac{\partial f}{\partial z} = 0 \quad , \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1, \ z = 0\right) \tag{1.5}$$

and $p^*(x,y)$ is a particular solution of the Poisson equation in two-dimensions

$$\frac{\partial^2 p^*}{\partial x^2} + \frac{\partial^2 p^*}{\partial y^2} = \frac{p(x,y)}{2\mu} \tag{1.6}$$

The problem is to determine the potential function $f(x,y,z)$ such that it satisfies (1.4) and (1.5).

Green and Sneddon [3] have examined the problem of a flat elliptical discontinuity opened by a uniform pressure, $p(x,y)$ = constant. In part of the work to follow, $p(x,y)$ is taken to be a linear function of x and/or y, i.e.,

$$p(x,y) = C_1 + C_2 x + C_3 y . \tag{1.7}$$

where c_j (j=1,2,3) are constants related to the intensity of the applied pressure. An exact solution is given for the case of $c_1 = c_3 = 0$ and $c_2 = -p_0/2\mu$. The same method of solution may be used for $c_1 = c_2 = 0$ and $c_3 = -p_0/2\mu$. An attempt has also been made to find $f(x,y,z)$ corresponding to a general expression of $p(x,y)$, say

$$p(x,y) = \sum_{n=0}^{m} \left(A_n x^{2n} + B_n y^{2n} \right) \tag{1.8}$$

The end results appear to be rather restricted in the sense that the corffi-cients A_n, B_n of the polynomial $p(x,y)$ are required to depend upon each other.

An alternative formulation of the problem is to determine the distribution of pressure necessary to preserve certain shape of the surfaces of discontinuity. This can be accomplished by specifying equal and opposite normal displacements, $q(x,y)$, on the region given by (1.1). For this class of problems, the boundary conditions are

$$\frac{\partial f}{\partial z} = - \frac{q(x,y)}{2(1-\nu)} \quad , \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, \ z = 0\right) \tag{1.9}$$

$$\frac{\partial f}{\partial z} = 0 \quad , \qquad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1,\ z = 0\right) \qquad (1.10)$$

where ν ie Poieeon'e ratio. Equatione (1.9) and (1.10) show that $f(x,y,z)$ is equivalent to the potential of a eimple layer distributed over an elliptical diek. The mase deneity of the diek ie proportional to $q(x,y)$. For this reason, eolutione of considerebly general in nature may be found. In fact, if the normal dieplacemente are given by a polynomial $q(x,y)$ of degree n in x^2 and y^2, then the integrale repreeenting $f(x,y,z)$ cen be evaluated in cloeed form. The only limitation for the ellipticel geometry is thet the degree of $q(x,y)$ must be equel to m+½, m being a poeitive integer. The circular region problem, however, ie free from euch a reetriction.

The problem of epecifying preseures and/or displacemente, equal in magnitude but opposite in sign, to the plenee of a flat ellipee may also be formuleted. Kessir and Sih [4] have shown that the the ekew-symmetric problem requires the knowledge of two hermonic functione $g(x,y,z)$ and $h(x,y,z)$. The simple ceee of en ellipticelly-shaped plane of diecontinuity, whose eurfecee ere eubjected to uniform eheer, wee diecuseed in [4]. Without further commente, it ie obvioue that a generel treetment of the skew-symmetric problem followe immediately from the work preeented in thie paper end in [4].

2. Equetione in Elestoetatics. Coneider the problem of finding etreesee end displecemente in an infinite elaetic eolid whoee continuity is interrupted by a void in the ehape of a plene ellipee. The epplied preesuree or displacements on the upper eurfece of thie ellipee are equal to those on the lower eurfece. Hence, the problem ie said to be symmetric with respect to the ellipticel region located, eey at z = o or in the xy-plane. Outeide of this region, the eheer etreeeee τ_{xz} end τ_{yz} muet vanish et z = o. These conditions mey be eatisfied by expressing the certeeien componente u, v, and w of the

displacement vector in terms of a single harmonic function f(x,y,z) as

$$u = (1-2\nu)\frac{\partial f}{\partial x} + z\frac{\partial^2 f}{\partial x \partial z}, \tag{2.1}$$

$$v = (1-2\nu)\frac{\partial f}{\partial y} + z\frac{\partial^2 f}{\partial y \partial z}, \tag{2.2}$$

$$w = -2(1-\nu)\frac{\partial f}{\partial z} + z\frac{\partial^2 f}{\partial z^2}. \tag{2.3}$$

The details are omitted here since they have already been given elsewhere [2]. In the usual way, the stress components τ_{xz}, τ_{yz}, and σ_{zz} are given by

$$\tau_{xz} = 2\mu z\frac{\partial^3 f}{\partial x \partial z^2}, \tag{2.4}$$

$$\tau_{yz} = 2\mu z\frac{\partial^3 f}{\partial y \partial z^2}, \tag{2.5}$$

$$\sigma_{zz} = 2\mu\left(-\frac{\partial^2 f}{\partial z^2} + z\frac{\partial^3 f}{\partial z^3}\right). \tag{2.6}$$

For the mere purpose of determining the potential function f(x,y,z), the remaining stress components σ_{xx}, σ_{yy}, and τ_{xy} are not needed.

3. Linearly Varying Pressure. Let the two sides of the elliptical plane of discontinuity be subjected to a pressure that varies linearly with the coordinate x over the region in (1.1). The disturbance of the discontinuity diminishes at large distance away from the origin. On the plane z = o, (2.3) and (2.6) yield

$$\frac{\partial^2 f}{\partial z^2} = -\frac{p_0 x}{2\mu}, \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1,\ z = 0\right) \tag{3.1}$$

$$\frac{\partial f}{\partial z} = 0, \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1,\ z = 0\right) \tag{3.2}$$

It is assumed here that the second and third derivatives of f(x,y,z) with respect to z are bounded in the limit as z approaches zero. These requirements will be met in the subsequent analysis.

The problem can be readily solved upon defining a new harmonic function $\phi(x,y,z)$ related to $f(x,y,z)$ by

$$\frac{\partial f}{\partial z} = x\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial x}, \tag{3.3}$$

where

$$\nabla^2 f(x,y,z) = 0, \qquad \nabla^2 \phi(x,y,z) = 0$$

and ∇^2 is the Laplacian operator in three dimensions. From (3.3), it can be shown that

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\phi) + \int_z^\infty z\frac{\partial^2 \phi}{\partial x^2}\,dz, \tag{3.4}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\phi) + \int_z^\infty z\frac{\partial^2 \phi}{\partial y\partial x}\,dz. \tag{3.5}$$

The limits of the integrals in (3.4) and (3.5) are introduced in such a way that the stresses and displacements vanish as z goes to infinity. In terms of the function $\phi(x,y,z)$, the displacements become

$$u = (1-2\nu)\frac{\partial}{\partial x}\left[x\phi + \int_z^\infty z\frac{\partial \phi}{\partial x}dz\right] + z\frac{\partial^2}{\partial x\partial z}\left[x\phi + \int_z^\infty z\frac{\partial \phi}{\partial x}dz\right], \tag{3.6}$$

$$v = (1-2\nu)\frac{\partial}{\partial y}\left[x\phi + \int_z^\infty z\frac{\partial \phi}{\partial x}dz\right] + z\frac{\partial^2}{\partial y\partial z}\left[x\phi + \int_z^\infty z\frac{\partial \phi}{\partial x}dz\right], \tag{3.7}$$

$$w = -2(1-\nu)\left[x\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial x}\right] + z\frac{\partial}{\partial z}\left[x\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial x}\right]. \tag{3.8}$$

Using (3.3)-(3.5), the stress components are

$$\tau_{xz} = 2\mu z\frac{\partial^2}{\partial x\partial z}\left(x\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial x}\right), \tag{3.9}$$

$$\tau_{yz} = 2\mu z\frac{\partial^2}{\partial y\partial z}\left(x\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial x}\right), \tag{3.10}$$

$$\sigma_{zz} = 2\mu\left[\frac{\partial \phi}{\partial x} - x\frac{\partial^2 \phi}{\partial z^2} + z\frac{\partial}{\partial z}\left(x\frac{\partial^2 \phi}{\partial z^2} - z\frac{\partial^2 \phi}{\partial x\partial z}\right)\right], \tag{3.11}$$

Thus, the boundary conditions (3.1) and (3.2) take the form

$$\frac{\partial\phi}{\partial x} - x\frac{\partial^2\phi}{\partial z^2} = \frac{p_0 x}{2\mu}, \qquad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}<1,\ z=0\right) \tag{3.12}$$

$$x\frac{\partial\phi}{\partial z} - z\frac{\partial\phi}{\partial x} = 0, \qquad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}>1,\ z=0\right) \tag{3.13}$$

Upon introducing a system of ellipsoidal coordinates, the position of a point can be determined by the three parameters ξ, η, ζ, the limits of variation of which are

$$\infty > \xi \geq 0 \geq \eta \geq -b^2 \geq \zeta \geq -a^2. \tag{3.14}$$

The cartesian coordinates x,y,z may be expressed in terms of these parameters by the relations

$$a^2(a^2-b^2)x^2 = (a^2+\xi)(a^2+\eta)(a^2+\zeta), \tag{3.15}$$

$$b^2(b^2-a^2)y^2 = (b^2+\xi)(b^2+\eta)(b^2+\zeta), \tag{3.16}$$

$$a^2b^2z^2 = \xi\eta\zeta. \tag{3.17}$$

In the plane z = o, the inside of the ellipse $x^2/a^2+y^2/b^2 = 1$ may be distinguished from the outside by setting $\xi = 0$ and $\eta = 0$, respectively. The boundary of the ellipse is identified by $\xi = \eta = 0$.

Since the function $\phi(x,y,z)$ is equivalent to the gravitational potential of a homogeneous elliptical disk at an exterior point x,y,z, the solution is given by [5]

$$\phi(x,y,z) = \frac{A}{2}\int_{\xi}^{\infty}\left[\frac{x^2}{a^2+s}+\frac{y^2}{b^2+s}+\frac{z^2}{s}-1\right]\frac{ds}{\sqrt{Q(s)}}, \tag{3.18}$$

where

$$Q(s) = s(a^2+s)(b^2+s). \tag{3.19}$$

Inserting (3.18) into (3.12) and (3.13), the constant A is found to be

$$A = \left(\frac{p_0 a^3}{4\mu}\right)\cdot\frac{k^2k'^2}{k'^2K(k)-E(k)}. \tag{3.20}$$

In (3.20), K(k) and E(k) are Legendre's complete elliptic integrals of the first and second kind, respectively, for the modulus

$$k^2 = 1 - \left(\frac{b}{a}\right)^2,$$

and $k'^2 = 1 - k^2$. Once $\phi(x,y,z)$ is known, the stresses and displacements at any point of the solid can be calculated in a straightforward manner.

For instance, the displacements u, v, and w in the plane z = o may be obtained readily from (3.6)-(3.8) which reduce to

$$u = (1-2\nu)\frac{\partial}{\partial x}\left(x\phi + \frac{\partial \psi}{\partial x}\right), \tag{3.21}$$

$$v = (1-2\nu)\frac{\partial}{\partial y}\left(x\phi + \frac{\partial \psi}{\partial x}\right), \tag{3.22}$$

$$w = -2(1-\nu)\, x \frac{\partial \phi}{\partial z} \tag{3.23}$$

in which

$$\psi(x,y,z) = \int_z^\infty z\, \phi(x,y,z)\, dz . \tag{3.24}$$

Introducing the contraction

$$\omega(s) = 1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{s}, \tag{3.25}$$

and substituting (3.18) into (3.24), the function $\psi(x,y,z)$ may be represented by the integral

$$\begin{aligned}\psi(x,y,z) &= -\frac{A}{2}\int_z^\infty z\left[\int_\xi^\infty \omega(s)\frac{ds}{\sqrt{Q(s)}}\right]dz \\ &= -\frac{A}{8}\int_\xi^\infty [\omega(s)]^2 \frac{s\,ds}{\sqrt{Q(s)}} .\end{aligned} \tag{3.26}$$

It follows that

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{A}{2}\int_\xi^\infty \omega(s)\cdot\frac{s\,ds}{(a^2+s)\sqrt{Q(s)}} - A x^2 \int_\xi^\infty \frac{s\,ds}{(a^2+s)^2\sqrt{Q(s)}}, \tag{3.27}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = -Axy\int_\xi^\infty \frac{s\,ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} . \tag{3.28}$$

The results of putting (3.18), (3.27), and (3.28) into (3.21)-(3.23) are

$$u = (1-2\nu)\cdot\frac{Aa^2}{2}\left[3x^2\int_\xi^\infty \frac{ds}{(a^2+s)^2\sqrt{Q(s)}} + y^2\int_\xi^\infty \frac{ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} - \int_\xi^\infty \frac{ds}{(a^2+s)\sqrt{Q(s)}}\right], \quad (3.29)$$

$$v = (1-2\nu)\cdot A a^2 x y \int_\xi^\infty \frac{ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}}, \quad (3.30)$$

$$w = -2(1-\nu)Ax \lim_{z\to 0}\left[z\int_\xi^\infty \frac{ds}{s\sqrt{Q(s)}}\right]. \quad (3.31)$$

The integrals in (3.29)-(3.31) may be evaluated in closed form. They are

$$\int_\xi^\infty \frac{ds}{(a^2+s)\sqrt{Q(s)}} = \frac{2}{a^3k^2}\left[u - E(u)\right], \quad (3.32)$$

$$\int_\xi^\infty \frac{ds}{(b^2+s)\sqrt{Q(s)}} = \frac{2}{a^3k^2k'^2}\left[E(u) - k'^2u - k^2\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u}\right], \quad (3.33)$$

and others of the same type. The quantity E(u) is

$$E(u) = \int_0^u \operatorname{dn}^2 t\, dt, \quad (3.34)$$

and snu, cnu, dnu are the Jacobian elliptic functions whose variable u should be distinguished from the x-component of the displacement vector. Here u is related to the ellipsoidal coordinate ξ by

$$\xi = a^2\operatorname{cn}^2u/\operatorname{sn}^2u = a^2(\operatorname{sn}^{-2}u - 1). \quad (3.35)$$

After some manipulations, the displacements for z = o are

$$u = \frac{(1-2\nu)A}{a^3k^2}\Big\{x^2\left[\left(1+\frac{2}{k^2}\right)u - 2\left(1+\frac{1}{k^2}\right)E(u) + \operatorname{sn}u\operatorname{cn}u\operatorname{dn}u\right]$$
$$- y^2\left[\frac{2}{k^2}u - \frac{1+k'^2}{k^2k'^2}E(u) + \frac{1}{k'^2}\operatorname{sn}u\operatorname{cn}u/\operatorname{dn}u\right] + a^2\left[E(u) - u\right]\Big\}, \quad (3.36)$$

$$v = \frac{(1-2\nu)A}{a^3k^2}\cdot xy\left[\left(1+\frac{1}{k'^2}\right)E(u) - 2u - \frac{k^2}{k'^2}\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u}\right], \quad (3.37)$$

$$w = -\frac{4(1-\nu)A}{ab^2}\, x \lim_{\zeta\to 0} \zeta\left[\frac{sn\,u\; dn\,u}{cn\,u} - E(u)\right]. \tag{3.38}$$

On the plane surface of the discontinuity, i.e., for $\xi = 0$, (3.36)-(3.38) may be simplified since

$$E(u) \to E(k)\;,\; u \to K(k)\;,\; \frac{sn\,u\; cn\,u}{dn\,u} \to 0\;,\quad as\;\; \xi \to 0.$$

Hence, the deformed shape of the ellipse $x^2/a^2+y^2/b^2 = 1$ is given by

$$(u)_{\xi=0} = \frac{(1-2\nu)A}{a^3k^4}\Big\{x^2\left[(2+k^2)K(k) - 2(1+k^2)E(k)\right] - y^2\left[2K(k) - (1+\tfrac{1}{k'^2})E(k)\right] + a^2k^2\left[E(k) - K(k)\right]\Big\}, \tag{3.39}$$

$$(v)_{\xi=0} = \frac{2(1-2\nu)A}{a^3k^4}\cdot xy\left[(1+\tfrac{1}{k'^2})E(k) - 2K(k)\right], \tag{3.40}$$

$$(w)_{\xi=0} = -\frac{4(1-\nu)Ax}{ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}. \tag{3.41}$$

The constant A has already been determined in (3.20).

Of interest is the intensification of the normal stress σ_{zz} in the neighborhood of the geometric discontinuity. It is the local elevation of σ_{zz} in the vicinity of the edge of the discontinuity ($\xi \to 0$, $\eta \to 0$) that controls the stability behavior [4] of the plane elliptical cavity. From (3.11)

$$\sigma_{zz} = 2\mu\left(\frac{\partial\phi}{\partial x} - x\frac{\partial^2\phi}{\partial z^2}\right), \quad z = 0 \tag{3.42}$$

Inserting (3.18) into (3.42) and integrating, the obtained result gives

$$(\sigma_{zz})_{\eta=0} = \frac{2Ax}{a^3k^2}\left\{u - E(u) - \frac{a^3k^2}{\sqrt{Q(\xi)}} - (1-\frac{a^2}{b^2})\left[E(u) - \frac{sn\,u\; cn\,u}{dn\,u}\right]\right\}_{\eta=0} \tag{3.43}$$

The expression (3.43) is valid only in the region outside of the ellipse $x^2/a^2+y^2/b^2 = 1$. To find σ_{zz} near the periphery of the elliptical boundary, the asymptotic value of ξ is required. By taking a vector $\underset{\sim}{\rho}$ normal to the curve that defines the ellipse with parametric equations

$$x = a \sin\phi \quad , \quad y = b \cos\phi \tag{3.44}$$

the expansion

$$\xi = \frac{2ab\rho}{(a^2\sin^2\phi + b^2\cos^2\phi)^{1/2}} + \cdots \tag{3.45}$$

is obtained for which terms of order higher than ρ have been neglected since ρ, the magnitude of $\underset{\sim}{\rho}$, has been assumed to be small in comparison with either a or b. In addition, it can be shown that

$$\zeta = -(a^2\sin^2\phi + b^2\cos^2\phi) \tag{3.46}$$

The limiting forms of ξ and ζ lead to

$$(\sigma_{zz})_{\eta=0} = -\frac{4\mu}{a^{1/2}b^{3/2}} \frac{A}{\sqrt{2\rho}} (a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{4}} \cos\phi + \cdots \quad , \quad \xi \to 0 \tag{3.47}$$

Denoting the coefficient of $1/\sqrt{2\rho}$ by k_1 and using (3.20) yield

$$k_1 = \frac{p_0 k^2 (ab)^{1/2}}{E(k) - k'^2 K(k)} (a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{4}} \cos\phi \tag{3.48}$$

This is known as the three-dimensional "stress-intensity factor" [4] the critical value of which governs the onset of unstable motion of the geometric discontinuity under consideration.

The applied pressure on the ellipse $x^2/a^2+y^2/b^2 = 1$ may also depend on x^{2n}, y^{2n}, where n is a positive integer. For n = 1, the function

$$f(x,y,z) = \frac{B}{2} \int_{\xi}^{\infty} [\omega(s)]^2 \frac{ds}{\sqrt{Q(s)}} \tag{3.49}$$

will satisfy the boundary conditions

$$\frac{\partial^2 f}{\partial z^2} = \frac{4B}{a^5 k^2} \Big\{ \frac{a^4 k^2}{b^2} E(k) - \Big[\Big(\frac{k^2}{k'^2} - 1\Big) E(k) + K(k) \Big] x^2 - \tag{3.50}$$

$$- \frac{1}{k'^2}\left[\left(\frac{k^2+1}{k'^2}\right)E(k) - K(k)\right] y^2 \Big\} \ , \quad \xi = 0$$

$$\frac{\partial f}{\partial z} = 0 \qquad , \qquad \eta = 0 \qquad (3.51)$$

However, the reault baaed on (3.49) will obviously be limited to the type of preasure diatribution aa indicated in (3.50). Similar obaervationa may be made for n greater than one.

4. Specification of Displacements. As mentioned earlier, when the region of discontinuity undergoes certain normal displacements, q(x,y), that are preacribed, the problem can in general be solved for the diatribution of the normal pressure which is initially unknown. The conditiona (1.9) and (1.10) suggest taking the function f(x,y,z) as the Newtonian potential of a simple layer of intensity q(x,y) diatributed over the plane region Ω in the shape of the ellipse $x^2/a^2+y^2/b^2 = 1$. The potential is

$$f(x,y,z) = \frac{1}{4\pi(1-\nu)} \iint_{\Omega} \frac{q(X,Y)}{R}\, dX\, dY \ , \qquad (4.1)$$

where R denotes

$$R = \left[(x-X)^2 + (y-Y)^2 + z^2\right]^{1/2} .$$

The function f(x,y,z) ia harmonic and ia continuous in the entire three-dimensional region excluding Ω . Furthermore, it vanishea for distances sufficiently far away from Ω since

$$f \to \frac{1}{R}\left[\frac{1}{4\pi(1-\nu)} \iint_{\Omega} q(X,Y)\, dX\, dY\right] \ , \quad \text{as} \quad R \to \infty$$

The normal derivative of f(x,y,z) ia diacontinuous for the transition from one side of Ω , say $z = 0^+$, to the other, $z = 0^-$, i.e.,

$$\left(\frac{\partial f}{\partial z}\right)_{z \to 0^\pm} = \begin{cases} \mp \dfrac{q(x,y)}{2(1-\nu)} \ , & \xi = 0 \qquad (4.2) \\[2ex] 0 \ , & \eta = 0 \qquad (4.3) \end{cases}$$

Hence, (1.9) and (1.10) are satisfied automatically when $f(x,y,z)$ is constructed from the potential of a simple layer.

For the present problem, it is more convenient to employ the symmetrical form of ellipsoidal coordinates ξ, η, ζ as given in (3.14)-(3.17). Proposed as a possible solution is the function [6]

$$f(x,y,z) = \int_{\xi}^{\infty} \lambda(\omega) \frac{ds}{\sqrt{Q(s)}} \tag{4.4}$$

The variable ω is defined by (3.25) or the equivalent form

$$\omega(s) = \frac{(s-\xi)(s-\eta)(s-\zeta)}{Q(s)}$$

and λ is a twice differentiable function in the interval (o,1) with finite one-sided derivatives at the boundary points of the interval. Differentiating (4.4) with respect to x renders

$$\frac{\partial f}{\partial x} = \int_{\xi}^{\infty} \frac{\partial \lambda(\omega)}{\partial x} \frac{ds}{\sqrt{Q(s)}} + \lambda(0) \frac{\partial I_0(\xi)}{\partial x} , \tag{4.5}$$

$$\frac{\partial^2 f}{\partial x^2} = \int_{\xi}^{\infty} \frac{\partial^2 \lambda(\omega)}{\partial x^2} \frac{ds}{\sqrt{Q(s)}} + 4 \frac{\lambda'(0)}{\sqrt{Q(\xi)}} \frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial x} + \lambda(0) \frac{\partial^2 I_0(\xi)}{\partial x^2} , \tag{4.6}$$

in which

$$I_0(\xi) = \int_{\xi}^{\infty} \frac{ds}{\sqrt{Q(s)}} ,$$

satisfies the harmonic equation $\nabla^2 I_0 = 0$. The remaining expressions of $\partial f/\partial y$, $\partial^2 f/\partial y^2$, and $\partial f/\partial z$, $\partial^2 f/\partial z^2$, may be obtained simply by permutation of the variable x in (4.5), (4.6) to y and z, respectively. As a result, the Laplacian of $f(x,y,z)$ can be writted as

$$\nabla^2 f(x,y,z) = \int_{\xi}^{\infty} \nabla^2 \lambda(\omega) \frac{ds}{\sqrt{Q(s)}} + 4 \frac{\lambda'(0)}{\sqrt{Q(\xi)}} , \tag{4.7}$$

with the knowledge that [7]

$$\frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial z} = 1 .$$

Now, computing the derivatives

$$\frac{\partial\lambda(\omega)}{\partial x} = -\frac{2x}{a^2+s}\lambda'(\omega),$$

$$\frac{\partial^2\lambda(\omega)}{\partial x^2} = -\frac{2}{a^2+s}\left[\lambda'(\omega) - \frac{2x^2}{a^2+s}\lambda''(\omega)\right],$$

and etc., the integrand, $\nabla^2\lambda(\omega)$, in (4.7) becomes

$$\nabla^2\lambda(\omega) = -\frac{2Q'(s)}{Q(s)}\lambda'(\omega) + 4\omega'(s)\lambda''(\omega) = 4\sqrt{Q(s)}\,\frac{d}{ds}\left[\frac{\lambda'(\omega)}{\sqrt{Q(s)}}\right]. \tag{4.8}$$

By means of (4.7) and (4.8), $f(x,y,z)$ is indeed a solution of the equation

$$\nabla^2 f(x,y,z) = 0.$$

There remains the determination of $\lambda(\omega)$ which may be evaluated from the boundary conditions (4.2) and (4.4). First of all, replacing x by z in (4.5) and using the relation

$$\frac{\partial\xi}{\partial z} = \frac{2zQ(\xi)}{\xi(\xi-\eta)(\xi-\zeta)},$$

the result is

$$\frac{\partial f}{\partial z} = -2\frac{(\xi\eta\zeta)^{\frac{1}{2}}}{ab}\int_\xi^\infty \lambda'(\omega)\frac{ds}{s\sqrt{Q(s)}} - 2\lambda(0)\frac{[\eta\zeta(a^2+\xi)(b^2+\xi)]^{\frac{1}{2}}}{ab(\xi-\eta)(\xi-\zeta)}. \tag{4.9}$$

While (4.9) satisfies (4.3) for $\eta = 0$, it reduces to an integral equation of the Abel type for $\xi = 0$ as dictated by (4.2). This will be shown in the next section.

5. <u>The Abel Integral</u>. Applying the boundary condition (4.2) for points of x, y inside $x^2/a^2+y^2/b^2 = 1$ and $z \to 0^+$, (4.9) may be written as

$$\frac{ab\,q(x,y)}{4(1-\nu)} = \lim_{\xi\to 0}\left[(\xi\eta\zeta)^{\frac{1}{2}}\int_\xi^\infty \lambda'(\omega)\frac{ds}{s\sqrt{Q(s)}}\right] + \frac{ab\,\lambda(0)}{(\eta\zeta)^{\frac{1}{2}}}. \tag{5.1}$$

The reduction of (5.1) to Abel's equation may be accomplished by changing the variable of integration from s to $\alpha = 1-\xi/s$ and by keeping in mind that ω is a function of s with the limiting form

$$\omega(s) \to \frac{\alpha\eta\zeta}{a^2b^2}, \quad \text{as} \quad \xi \to 0$$

where

$$\eta\zeta \to a^2b^2 Z, \text{ and } Z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}. \tag{5.2}$$

Under these considerations, (5.1) reduces to

$$\frac{ab\,q(Z)}{4(1-\nu)} = Z^{\frac{1}{2}}\int_0^1 \lambda'(\alpha Z)\frac{d\alpha}{\sqrt{1-\alpha}} + \frac{\lambda(0)}{Z^{\frac{1}{2}}} . \tag{5.3}$$

An additional change of variable to $\beta = \alpha Z$ gives the Abel integral equation [6]

$$\frac{ab\,q(Z)}{4(1-\nu)} = \int_0^Z \frac{\lambda'(\beta)\,d\beta}{\sqrt{Z-\beta}} + \frac{\lambda(0)}{Z^{1/2}} . \tag{5.4}$$

In order to solve for $\lambda(\omega)$, both sides of (5.4) are integrated from o to ω by $dZ/\sqrt{\omega - Z}$. Hence, (5.4) becomes

$$\int_0^\omega \frac{\lambda(0)\,dZ}{Z^{1/2}\sqrt{\omega-Z}} + \int_0^\omega \frac{dZ}{\sqrt{\omega-Z}}\int_0^Z \frac{\lambda'(\beta)\,d\beta}{\sqrt{Z-\beta}} = \frac{ab}{4(1-\nu)}\int_0^\omega \frac{q(Z)\,dZ}{\sqrt{\omega-Z}} . \tag{5.5}$$

The integrals on the left hand side of (5.5) may be evaluated without difficulty the first of which is

$$\int_0^\omega \frac{\lambda(0)\,dZ}{Z^{1/2}\sqrt{\omega-Z}} = 2\int_0^{\frac{\pi}{2}} \lambda(0)\,d\theta = \pi\lambda(0) . \tag{5.6}$$

The order of integration of the second integral

$$\int_0^\omega \frac{dZ}{\sqrt{\omega-Z}}\int_0^Z \frac{\lambda'(\beta)\,d\beta}{\sqrt{Z-\beta}} \tag{5.7}$$

may be interchanged since the integrands possess weak singularities. Application of the Dirichlet formula [7] gives

$$\int_0^\omega \lambda'(\beta)\,d\beta\int_\beta^\omega \frac{dZ}{\sqrt{(\omega-Z)(Z-\beta)}} = \pi\int_0^\omega \lambda'(\beta)\,d\beta = \pi\left[\lambda(\omega)-\lambda(0)\right] . \tag{5.8}$$

Substituting both (5.6) and (5.8) into (5.5) yields the solution of (5.4) as

$$\lambda(\omega) = \frac{ab}{4\pi(1-\nu)}\int_0^\omega \frac{q(Z)\,dZ}{\sqrt{\omega-Z}} . \tag{5.9}$$

Thus, (5.9) solves the problem of prescribing normal displacements q(Z) on an elliptically-shaped plane of discontinuity.

6. Theorem. On the basis of the results obtained in the previous section,

the following theorem may be established:

"Let the normal displacements, $q(x,y)$, on the plane surfaces of the ellipse $x^2/a^2+y^2/b^2 = 1$ be given by $(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})^{\frac{1}{2}} Q_n(x^2,y^2)$, where $Q_n(x^2,y^2)$ is a polynomial of degree n in x^2, y^2. Then, the normal pressure acting over the ellipse is also a polynomial, $P_n(x^2,y^2)$, of the same degree in x^2,y^2."

Now, suppose that

$$q(x,y) = Z^{\frac{1}{2}} \sum_{j=0}^{n} C_j Z^j . \tag{6.1}$$

It suffices to prove the theorem by taking $Q_n(x^2,y^2)$ in the form of (6.1). The coefficients C_j $(j=0,1,\cdots,n)$ are constants and Z is given by (5.2). Putting (6.1) into (5.9) and carrying out the integration yields

$$\lambda(\omega) = \frac{ab}{4(1-\nu)\sqrt{\pi}} \sum_{j=0}^{n} \frac{\Gamma(j+\frac{3}{2})}{(j+1)!} C_j [\omega(s)]^{j+1} , \tag{6.2}$$

in which $\Gamma(n)$ is the customary Gamma function. Before the normal pressure σ_{zz} can be calculated, the harmonic function $f(x,y,z)$ must be obtained from (4.4):

$$f(x,y,z) = \frac{ab}{4(1-\nu)\sqrt{\pi}} \sum_{j=0}^{n} \frac{\Gamma(j+\frac{3}{2})}{(j+1)!} C_j \int_{\xi}^{\infty} [\omega(s)]^{j+1} \frac{ds}{\sqrt{Q(s)}} . \tag{6.3}$$

In the plane $z = 0$, (2.6) simplifies to

$$(\sigma_{zz})_{z=0} = 2\mu \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)_{z=0} . \tag{6.4}$$

Adopting the notation

$$\gamma = \begin{cases} 0 , & \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, z=0 \right) \\ \xi , & \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1, z=0 \right) \end{cases} \tag{6.5}$$

and using (6.3), it is found that

$$(\sigma_{zz})_{z=0} = - \frac{ab\mu}{(1-\nu)\sqrt{\pi}} \sum_{j=0}^{n} \frac{\Gamma(j+\frac{3}{2})}{j!} C_j \left\{ \int_{\gamma}^{\infty} \left(\frac{1}{a^2+s} + \frac{1}{b^2+s} \right) [\omega_0(s)]^j \frac{ds}{\sqrt{Q(s)}} \right. \tag{6.6}$$

$$-2j\int_{\gamma}^{\infty}\left[\left(\frac{x}{a^2+s}\right)^2+\left(\frac{y}{b^2+s}\right)^2\right]\left[\omega_0(s)\right]^{j-1}\frac{ds}{\sqrt{Q(s)}}\ \Big\}\ ,$$

where

$$\omega_0(s)=\left[\omega(s)\right]_{z=0}=1-\frac{x^2}{a^2+s}-\frac{y^2}{b^2+s}\ .$$

The integrands containing $[\omega_0(a)]^n$, $(n=1,2,\cdots)$, may be expanded in terms of x^2, y^2 to give

$$(\sigma_{zz})_{z=0}=A_0+A_1x^2+A_2y^2+\cdots=P_n(x^2,y^2)\ , \tag{6.7}$$

in which A_j $(j=0,1,2,\cdots,n)$ are

$$A_0=B_0\int_{\gamma}^{\infty}\left(\frac{1}{a^2+s}+\frac{1}{b^2+s}\right)\frac{ds}{\sqrt{Q(s)}}\ ,$$

$$A_1=B_1\int_{\gamma}^{\infty}\frac{ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}}+B_2\int_{\gamma}^{\infty}\frac{ds}{(a^2+s)^2\sqrt{Q(s)}}\ ,$$

$$A_2=B_1\int_{\gamma}^{\infty}\frac{ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}}+B_2\int_{\gamma}^{\infty}\frac{ds}{(b^2+s)^2\sqrt{Q(s)}}\ ,$$

and so on. The constants B_0, $\cdots$, B_2 are related to C_j $(j=0,1,2)$ as follows:

$$B_0=-\frac{ab\mu}{(1-\nu)\sqrt{\pi}}\sum_{j=0}^{n}\frac{\Gamma(j+\frac{3}{2})}{j!}C_j\ ,\quad B_1=\frac{ab\mu}{(1-\nu)\sqrt{\pi}}\sum_{j=1}^{n}\frac{\Gamma(j+\frac{3}{2})}{(j-1)!}C_j\ ,\quad B_2=3B_1\ .$$

From (6.7), $(\sigma_{zz})_{z=0}$ is seen to be a polynomial of degree n in x^2, y^2 whose coefficients depend on

$$\int_{\gamma}^{\infty}\frac{ds}{(a^2+s)^{n+1}\sqrt{Q(s)}}\ ,\ \int_{\gamma}^{\infty}\frac{ds}{(a^2+s)^{n+1}(b^2+s)^{n+1}\sqrt{Q(s)}}\ ,\ \int_{\gamma}^{\infty}\frac{ds}{(b^2+s)^{n+1}\sqrt{Q(s)}}\ ,\quad n=0,1,\cdots$$

These integrals can be reduced to the complete elliptic integrals of the first and second kind with modulus $k^2=1-(b/a)^2$ for $\xi=0$.

The aforementioned theorem also applies to the skew-symmetric problem of specifying displacements u, v on the elliptical plane of discontinuity. This is mainly because the boundary conditions on u, v are the same as those described by (4.2) and (4.3) since Kassir and Sih [4] have already shown that for the skew-symmetric case the displacements are

$$u = -2(1-\nu)\frac{\partial g}{\partial z} + z\frac{\partial G}{\partial x},$$

$$v = -2(1-\nu)\frac{\partial h}{\partial z} + z\frac{\partial G}{\partial y},$$

$$w = -(1-2\nu)G + z\frac{\partial G}{\partial z},$$

where

$$G = \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y},$$

and

$$\nabla^2 g(x,y,z) = 0, \qquad \nabla^2 h(x,y,z) = 0.$$

For this problem, the shear stresses τ_{xz}, τ_{yz} will be polynomials of degree n in x^2, y^2 if the displacements u, v are represented by polynomials of the form (6.1).

In the two-dimensional case, Sneddon [1] has observed that the prescribed shape of a slit in an isotropic medium and the resulting surface pressure may be represented by polynomials of the same degree in one variable. Galin [8] has considered the three-dimensional problem of an elliptical punch pressing against a semi-infinite elastic solid whose surface outside the area of contact is free from tractions. The frictional forces acting between the punch and the semi-infinite body are neglected. He showed that if the base profile of the punch is given by a polynomial of degree n in x, y, then the pressure acting over the punch is another polynomial. Galin's proof is based upon the properties of Lamé triple product, which is different from that given in the present paper.

7. The Symmetric Problem. A sufficiently general description of the shape of the elliptical plane of discontinuity is

$$q(Z) = q_0 Z^n, \tag{7.1}$$

where q_0 denotes the amplitude of the normal displacement. The restriction on the exponent n becomes apparent when the functions

$$\lambda(\omega) - \frac{abq_0}{4\pi(1-\nu)}\int_0^\omega \frac{Z^n dZ}{\sqrt{\omega - Z}} = \frac{abq_0}{4(1-\nu)\sqrt{\pi}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}[\omega(s)]^{n+\frac{1}{2}}, \quad n > -1, \tag{7.2}$$

and

$$f(x,y,z) = \frac{abq_0\Gamma(n+1)}{4(1-\nu)\sqrt{\pi}\,\Gamma(n+\frac{3}{2})}\int_\xi^\infty [\omega(s)]^{n+\frac{1}{2}}\frac{ds}{\sqrt{Q(s)}}, \quad n > -1, \tag{7.3}$$

are derived from (5.9) and (4.4), respectively. The normal pressure distribution required to support the shape (7.1) can be obtained from (6.4) and (7.3). For the range of n between 1/2 and 3/2, σ_{zz} in the plane z = o is

$$(\sigma_{zz})_{z=0} = -\frac{ab\mu q_0(n+\frac{1}{2})\Gamma(n+1)}{(1-\nu)\sqrt{\pi}\,\Gamma(n+\frac{3}{2})}\left[I_1 + I_2 + x\frac{\partial I_1}{\partial x} + y\frac{\partial I_2}{\partial y}\right], \quad \frac{1}{2} \le n < \frac{3}{2}, \tag{7.4}$$

where I_1, I_2 represent the integrals

$$I_1 = \int_\gamma^\infty \frac{[\omega_0(s)]^{n-\frac{1}{2}}}{(a^2+s)}\frac{ds}{\sqrt{Q(s)}}, \quad I_2 = \int_\gamma^\infty \frac{[\omega_0(s)]^{n-\frac{1}{2}}}{(b^2+s)}\frac{ds}{\sqrt{Q(s)}}$$

and γ is defined by (6.5). If n ≥ 3/2, the expression for $(\sigma_{zz})_{z=0}$ is less complicated:

$$(\sigma_{zz})_{z=0} = -\frac{ab\mu q_0(n+\frac{1}{2})\Gamma(n+1)}{(1-\nu)\sqrt{\pi}\,\Gamma(n+\frac{3}{2})}\left\{\int_\gamma^\infty \left(\frac{1}{a^2+s} + \frac{1}{b^2+s}\right)[\omega_0(s)]^{n-\frac{1}{2}}\frac{ds}{\sqrt{Q(s)}} - (2n-1)\int_\gamma^\infty \left[\left(\frac{x}{a^2+s}\right)^2 + \left(\frac{y}{b^2+s}\right)^2\right][\omega_0(s)]^{n-\frac{3}{2}}\frac{ds}{\sqrt{Q(s)}}\right\}, \quad n \ge \frac{3}{2}. \tag{7.5}$$

In general, the integrals appearing in (7.4) and (7.5) can be evaluated for n = m + ½, where m = o, 1, 2, ..., . In the degenerate case of a circular region of discontinuity, solutions may be found for any value of n.

8. Elliptical Region for n = 1/2 and n = 3/2. A special case of (7.4) is n = 1/2, which corresponds to the problem of a plane elliptical cut opened by a normal pressure, $(\sigma_{zz})_{z=0}$ = constant. The opening of the cut is ellipsoidal given by

$$q(Z) = q_0 Z^{\frac{1}{2}}.$$

This problem was formulated by Green ans Sneddon [3]. In a later paper, Kassir and Sih [4] pointed out that the stresses at the border of the cut are singular of the order of $1/\sqrt{\rho}$. The distance ρ is measured in a plane perpendicular to the boundary of the ellipse $x^2/a^2+y^2/b^2 = 1$.

Therefore, it is natural to inquire whether cuts of a shape other than ellipsoidal are possible and, if they exist, to determine the pressure necessary to preserve their shape. Consider the case

$$q(Z) = q_0 Z^{3/2}. \tag{8.1}$$

For n = 3/2, (7.5) specializes to

$$(\sigma_{zz})_{z=0} = -\frac{3ab\mu q_0}{4(1-\nu)}\left[\int_\gamma^\infty\left(\frac{1}{a^2+s}+\frac{1}{b^2+s}\right)\frac{ds}{\sqrt{Q(s)}} - 3x^2\int_\gamma^\infty\frac{ds}{(a^2+s)^2\sqrt{Q(s)}}\right.$$
$$\left. -3y^2\int_\gamma^\infty\frac{ds}{(b^2+s)^2\sqrt{Q(s)}} - (x^2+y^2)\int_\gamma^\infty\frac{ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}}\right]. \tag{8.2}$$

The integrals in (8.2) may be evaluated in a manner similar to those shown in (3.32) and (3.33).

The normal pressure inside of the elliptical region, $\gamma = 0$, is

$$(\sigma_{zz})_{z=0} = -\frac{3b\mu q_0}{2(1-\nu)a^2k^2}\left\{\left(\frac{k}{k'}\right)^2E(k) - \left(\frac{x}{b}\right)^2\left[k'^2K(k)+(k^2-k'^2)E(k)\right]\right.$$
$$\left. -\left(\frac{y}{b}\right)^2\left[\frac{1+k^2}{k'^2}E(k)-K(k)\right]\right\}. \tag{8.3}$$

When a = b and E(k) = K(k) = $\pi/2$, (8.3) gives the internal pressure for a penny-shaped cut of radius a, i.e.,

$$(\sigma_{zz})_{z=0} = -\frac{3\mu\pi}{4(1-\nu)}\left(\frac{q_0}{a}\right)\left[1-\frac{3}{2}\left(\frac{r}{a}\right)^2\right], \quad r<a \tag{8.4}$$

The sign of $(\sigma_{zz})_{z=0}$ changes from negative to positive at the ratio of $(r/a) \simeq 0.8$. This means that both compressive and tensile stresses must be applied to the surfaces of the cut in order to produce the shape (8.1).

Outside of the region of the ellipse, $(\sigma_{zz})_{z=0}$ may be found by setting $\gamma = \xi$ in (8.2):

$$(\sigma_{zz})_{\eta=0} = -\frac{3\mu}{2(1-\nu)}\left(\frac{q_0}{b}\right)\left\{E(u) - \frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u} - \right. \tag{8.5}$$

$$-\left(\frac{x}{ak}\right)^2\left[k'^2 u + (1-2k'^2)E(u) + \frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u}\left(k'^2\,\mathrm{dn}^2 u-1\right)\right]$$

$$-\left(\frac{y}{ak}\right)^2\left[\left(\frac{k^2}{k'^2}-1\right)E(u) - u - \frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u}\left(\frac{1}{\mathrm{dn}\,u} + 2\frac{k^2}{k'^2}-1\right)\right]\Bigg\}\,.$$

In contrast to the singular solution for $n = 1/2$, the stresses remain finite on the boundary of the elliptical cut. In fact, in the limit as $\xi \to 0$, (8.5) is of the order of $\rho^{1/2}$. The same behavior is observed for the particular case of $a = b$ since (8.5) reduces to

$$(\sigma_{zz})_{z=0} = -\frac{3\mu}{2(1-\nu)}\left(\frac{q_0}{a}\right)\left\{\left[1-\frac{3}{2}\left(\frac{r}{a}\right)^2\right]\sin^{-1}\left(\frac{a}{r}\right) + \frac{3}{2}\left[\left(\frac{r}{a}\right)^2-1\right]^{\frac{1}{2}}\right\}, \quad r>a, \tag{8.6}$$

where $(\sigma_{zz})_{z=0}$ goes to infinity as $O(1/r^3)$. When r approaches a, the circular boundary, the normal pressure

$$(\sigma_{zz})_{z=0} = \frac{3\mu\pi}{8(1-\nu)}\left(\frac{q_0}{a}\right), \quad r=a \tag{8.7}$$

is found to be non-singular.

It should be mentioned that (8.4) and (8.6) may be obtained directly from the method of Hankel transform for solving axially symmetric problems [9].

<u>9. Circular Region for Arbitrary n</u>. The theory of potential functions may also be applied expediently to a class of problems involving penny-shaped planes of discontinuities. For $a = b$, (7.4) can be solved in general without imposing restrictions on the exponent n as $I_1 = I_2 = I$ and

$$(\sigma_{zz})_{z=0} = -\frac{a^2\mu\, q_0\,(n+\frac{1}{2})\,\Gamma(n+1)}{(1-\nu)\sqrt{\pi}\,\Gamma(n+\frac{3}{2})}\left[2I + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)I\right]. \tag{9.1}$$

The integral I is

$$I = \int_{\gamma}^{\infty} \frac{[(s-\xi)(s-\eta)]^{n-\frac{1}{2}}\,ds}{s^n\,(a^2+s)^{n+\frac{3}{2}}}, \tag{9.2}$$

and the symbol γ stands for

$$\gamma = \begin{cases} 0, & r<a \\ r^2-a^2, & r>a\,. \end{cases} \tag{9.3}$$

Making the substitutions

$$\sigma^2 = \frac{s-\xi}{s-\eta}, \qquad \epsilon^2 = \frac{a^2+\eta}{a^2+\xi},$$

in (9.2), the integral I becomes

$$I = \frac{2(\xi-\eta)^{2n}}{(a^2+\xi)^{n+\frac{3}{2}}} \int_0^1 \frac{\sigma^{2n}\sqrt{1-\sigma^2}\, d\sigma}{(\xi-\sigma^2\eta)^n (1-\epsilon^2\sigma^2)^{n+\frac{3}{2}}} . \tag{9.4}$$

The determination of $(\sigma_{zz})_{z=0}$ for $r < a$ and $r > a$ will be carried out seperately.

Inside of the region $x^2+y^2 = a^2$ and for $z = 0$, the ellipsoidal coordinates ξ, η take the values

$$\xi = 0, \qquad \eta = r^2 - a^2,$$

and $\epsilon^2 = k^2 = (r/a)^2$. Hence, (9.4) reduces to the standard Legendre elliptic integral

$$(I)_{\xi=0} = \frac{2k'^{2n}}{a^3} \int_0^1 \frac{\sqrt{1-\sigma^2}\, d\sigma}{(1-k^2\sigma^2)^{n+\frac{3}{2}}}, \tag{9.5}$$

where

$$k'^2 = 1 - \left(\frac{r}{a}\right)^2.$$

A further substitution of $\sigma = \mathrm{sn}u$ leads to

$$(I)_{\xi=0} = \frac{2}{a^3}\left(\frac{k'^{2n}}{k^2}\right)\left(J_{2n} - k'^2 J_{2n+2}\right). \tag{9.6}$$

In (9.6), J_m represents the integral

$$J_m = \int_0^K (nd\, u)^m du, \tag{9.7}$$

whose values for $m = 0, 1, 2, \cdots$, are given by [10]

$$J_0 = K(k), \quad J_1 = \frac{\pi}{2k'}, \quad J_2 = \frac{E(k)}{k'^2}, \quad J_3 = \frac{(2-k^2)\pi}{4k'^3}, \cdots, \tag{9.8}$$

and the recurrence formulas

$$J_{2m+2} = \frac{1}{(2m+1)k'^2}\left[2m(2-k^2)J_{2m} + (1-2m)J_{2m-2}\right], \tag{9.9}$$

$$J_{2m+3} = \frac{1}{2(m+1)k'^2}\left[(2m+1)(2-k^2)J_{2m+1} - 2m J_{2m-1}\right]. \tag{9.10}$$

In order to complete the calculation of $(\sigma_{zz})_{z=0}$ for $\xi = 0$, it is necessary to find from (9.2) the quantity

$$\left[\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)I\right]_{\xi=0} = 2r^2\left(\frac{\partial I}{\partial \eta}\right)_{\xi=0}. \tag{9.11}$$

Some lengthy algebra gives

$$\left(\frac{\partial I}{\partial \eta}\right)_{\xi=0} = -\frac{k'^{2n}}{a^5}\left[\frac{2n}{k'^2}\int_0^1 \frac{\sqrt{1-\sigma^2}\,d\sigma}{(1-k^2\sigma^2)^{n+\frac{3}{2}}} - (2n+3)\int_0^1 \frac{\sigma^2\sqrt{1-\sigma^2}\,d\sigma}{(1-k^2\sigma^2)^{n+\frac{5}{2}}}\right]$$

$$= -\frac{k'^{2n}}{a^5 k^2}\left\{\frac{2n}{k'^2}\left(J_{2n} - k'^2 J_{2n+2}\right) + \left(\frac{2n+3}{k^2}\right)\left[J_{2n} - (1+k'^2)J_{2n+2} + k'^2 J_{2n+4}\right]\right\}, \tag{9.12}$$

where J_m is given by (9.7). Using (9.1), the internal pressure, $(\sigma_{zz})_{z=0}$, in the region $r < a$ takes the form

$$(\sigma_{zz})_{\xi=0} = -\frac{2\mu(n+\frac{1}{2})\Gamma(n+1)}{(1-\nu)\sqrt{\pi}\,\Gamma(n+\frac{3}{2})}\left(\frac{q_0}{a}\right)\left(\frac{k'^{2n}}{k^2}\right)\left[-\left(1+\frac{2n}{k'^2}\right)J_{2n} + (3+4n+k'^2)J_{2n+2} - (2n+3)k'^2 J_{2n+4}\right], \tag{9.13}$$

which is valid for arbitrary value of n. Three special cases of (9.13) will be considered by application of (9.8)-(9.10).

<u>Case 9.1(a)</u>. The necessary normal pressure to maintain the shape

$$q(r) = q_0\left[1-\left(\frac{r}{a}\right)^2\right]^{\frac{1}{2}}$$

may be computed from (9.13) by setting $n = 1/2$ as

$$(\sigma_{zz})_{\xi=0} = -\frac{\mu}{1-\nu}\left(\frac{q_0}{a}\right)\left(\frac{k'}{k^2}\right)\left[-\left(1+\frac{1}{k'^2}\right)J_1 + (5+k'^2)J_3 - 4k'^2 J_5\right]$$

$$= -\frac{\mu\pi}{2(1-\nu)}\left(\frac{q_0}{a}\right), \quad r < a. \tag{9.14}$$

This is in agreement with Sneddon's answer [9].

<u>Case 9.2(a)</u>. If $n = 1$, (9.13) simplifies to

$$(\sigma_{zz})_{\xi=0} = -\frac{4\mu}{(1-\nu)\pi}\left(\frac{q_0}{a}\right)\left(\frac{k'}{k}\right)^2\left[-\left(1+\frac{2}{k'^2}\right)J_2 + (7+k'^2)J_4 - 5k'^2 J_6\right]$$

$$= \frac{4\mu}{(1-\nu)\pi}\left(\frac{q_0}{a}\right)\left[K\left(\frac{r}{a}\right) - 2E\left(\frac{r}{a}\right)\right], \quad r < a. \tag{9.15}$$

At first sight, (9.15) appears to be different from the solution published by Sneddon [9] for the same problem, i.e.,

$$(\sigma_{zz})_{z=0} = -\frac{4\mu}{(1-\nu)\pi}\left(\frac{q_0}{a}\right)\left[\frac{1+(r/a)^2}{1+(r/a)} K(k^*) - K\left(\frac{r}{a}\right) - \left(1+\frac{r}{a}\right)E(k^*)\right], \tag{9.16}$$

in which

$$k^{*2} = \frac{4ar}{(a+r)^2}.$$

However, by means of the Landen transformation, the complete elliptic integrals in (9.16) can be written as

$$\left(1+\frac{r}{a}\right)E(k^*) = 2E\left(\frac{r}{a}\right) - \left[1-\left(\frac{r}{a}\right)^2\right]K\left(\frac{r}{a}\right), \tag{9.17}$$

$$K(k^*) = \left(1+\frac{r}{a}\right)K\left(\frac{r}{a}\right). \tag{9.18}$$

The identities (9.17), (9.18) show that (9.16) and (9.15) are indeed equivalent.

<u>Case 9.3(a)</u>. In the case of n = 3/2, (8.4) is recovered from (9.13) since

$$(\sigma_{zz})_{\xi=0} = -\frac{3\mu}{2(1-\nu)}\left(\frac{q_0}{a}\right)\left(\frac{k'^3}{k^2}\right)\left[-\left(1+\frac{3}{k'^2}\right)J_3 + (9+k'^2)J_5 - 6k'^2 J_7\right]$$

$$= -\frac{3\mu\pi}{4(1-\nu)}\left(\frac{q_0}{a}\right)\left[1-\frac{3}{2}\left(\frac{r}{a}\right)^2\right]. \tag{9.18}$$

In the same fashion, $(\sigma_{zz})_{z=0}$ outside of the circular plane of discontinuity may be obtained by letting

$$\left[\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)I\right]_{\eta=0} = 2r^2\left(\frac{\partial I}{\partial \xi}\right)_{\eta=0}, \tag{9.19}$$

and

$$(I)_{\eta=0} = \frac{2k'^{2n}}{r^3}\int_0^1 \frac{\sigma^{2n}\sqrt{1-\sigma^2}\,d\sigma}{(1-k^2\sigma^2)^{n+\frac{3}{2}}}, \tag{9.20}$$

which is found from (9.4) for $\eta = 0$, $\epsilon = k = (a/r)$, and $\xi = r^2 - a^2$. The substitution σ = snu allows (9.20) to be

$$(I)_{\eta=0} = \frac{2k'^{2n}}{r^3 k^2}\left(M_{2n} - k'^2 N_{2n}\right), \quad k = \frac{a}{r}, \tag{9.21}$$

where

$$M_m = \int_0^K \left(\frac{\mathrm{sn}\,u}{\mathrm{dn}\,u}\right)^m du \,, \tag{9.22}$$

$$N_{2n} = \int_0^K \frac{(\mathrm{sn}\,u)^{2n}}{(\mathrm{dn}\,u)^{2n+2}}\, du \,, \tag{9.23}$$

The integrals (9.22) can be expressed in terms of the complete elliptic integrals and elementary functions in the following manner [10] :

$$M_1 = \frac{\sin^{-1}(k) - \sin^{-1}(2kk')}{k\,k'} \,,\quad M_2 = \frac{E(k) - k'^2 K(k)}{k^2 k'^2} \,,\quad M_3 = \frac{M_1}{2kk'} \,, \cdots . \tag{9.24}$$

The recurrence relationships are

$$M_{2m+2} = \frac{1}{(2m+1)\,k^2 k'^2}\left[2m(2k^2-1)M_{2m} + (2m-1)M_{2m-2}\right], \tag{9.25}$$

$$M_{2m+3} = \frac{1}{2(m+1)\,k^2 k'^2}\left[(2m+1)(2k^2-1)\,M_{2m+1} + 2m\,M_{2m}\right]. \tag{9.26}$$

In addition, the integral (9.23) admits the relations

$$\begin{aligned} N_0 &= \frac{E(k)}{k'^2} \,,\quad N_1 = \frac{1}{3k^2k'^4}\left[(1+k^2)E(k) - k'^2K(k)\right], \cdots, \\ N_{m+2} &= \frac{1}{(5+2m)\,k^2 k'^2}\left[(2m+1)\,N_m + 2\,(3k^2 + 2mk^2 - m - 1)\,N_{m+1}\right], \end{aligned} \tag{9.27}$$

in which

$$N_m = \int_0^K \frac{(\mathrm{sn}\,u)^{2m}}{(\mathrm{dn}\,u)^{2m+2}}\, du \,.$$

If the exponent n = m/2, m being an integer, then (9.20) is reducible to elementary functions. Finally, the normal pressure for r>a is deduced:

$$\begin{aligned}(\sigma_{zz})_{\eta=0} = -\frac{2\mu(n+\frac{1}{2})\Gamma(n+1)}{(1-\nu)\pi\,\Gamma(n+\frac{3}{2})}\left(\frac{q_0}{r}\right) k'^{2n}\Big[2n\Big(\frac{1}{k'^2} - \frac{1}{2n} \qquad {}_{n} - k'^2 N_{2n}\Big) \\ - (2n+3)\,k^2\left(M_{2n+2} - k'^2 N_{2n+2}\right)\Big] .\end{aligned} \tag{9.28}$$

The values of $(\sigma_{zz})_{\eta=0}$ that correspond to n = 1/2, 1, 3/2 are as follows:

Case 9.1(b). Instead of using (9.28), it is simpler in the case of n=½ to compute $(\sigma_{zz})_{\eta=0}$ directly from (9.1)

$$(\sigma_{zz})_{\eta=0} = -\frac{\mu}{1-\nu}\left(\frac{q_0}{r}\right) k^2 k' \left[\left(\frac{1}{k'^2}-2\right)\int_0^1 \frac{\sigma\sqrt{1-\sigma^2}\,d\sigma}{(1-k^2\sigma^2)^2} - 4k^2\int_0^1 \frac{\sigma^3\sqrt{1-\sigma^2}\,d\sigma}{(1-k^2\sigma^2)^3}\right]. \tag{9.29}$$

These integrals are elementary giving

$$(\sigma_{zz})_{\eta=0} = -\frac{\mu}{1-\nu}\left(\frac{q_0}{r}\right)\left[\sin^{-1}\left(\frac{a}{r}\right) - \frac{1}{\sqrt{\left(\frac{r}{a}\right)^2-1}}\right], \quad r>a \tag{9.30}$$

which checks with that obatined by Sneddon [9].

Case 9.2(b). Letting n = 1 in (9.28) and carrying out the algebra render

$$(\sigma_{zz})_{\eta=0} = \frac{4\mu}{(1-\nu)\pi}\left(\frac{q_0}{a}\right)\left[\left(2\frac{r}{a}-\frac{a}{r}\right)K\left(\frac{a}{r}\right) - 2\frac{r}{a}E\left(\frac{a}{r}\right)\right], \quad 1<\frac{r}{a}<\infty, \tag{9.31}$$

which is O(1/r) for large r. At the boundary points of r = a, $(\sigma_{zz})_{\eta=0}$ is unbounded.

Case 9.3(b). The limiting form of (9.28) for n = 3/2 corresponds precisely to (8.6) which has already been discussed.

10. Conclusion. The problem of finding stresses and displacements in an elastic solid with geometric discontinuities has been reduced to the classical boundary problem of potential theory. Harmonic functions are developed for the case of an elliptically-shaped plane of discontinuity whose faces are subjected to pressures and/or displacements. While the present paper is primarily concerned with the calculation of the normal displacement w and the corresponding pressure σ_{zz}, the remaining displacements and stresses may be obtained from (2.1), (2.2), (2.4), and (2.5) without difficulty. Furthermore, with the knowledge of f(x,y,z), the stress components [2]

$$\sigma_{xx} = 2\mu\left(\frac{\partial^2 f}{\partial x^2} + 2\nu\frac{\partial^2 f}{\partial y^2} + z\frac{\partial^3 f}{\partial z\,\partial x^2}\right), \tag{10.1}$$

$$\sigma_{yy} = 2\mu\left(\frac{\partial^2 f}{\partial y^2} + 2\nu\frac{\partial^2 f}{\partial x^2} + z\frac{\partial^3 f}{\partial z\,\partial y^2}\right), \tag{10.2}$$

$$\tau_{xy} = 2\mu\left[(1-2\nu)\frac{\partial^2 f}{\partial x\,\partial y} + z\frac{\partial^3 f}{\partial x\,\partial y\,\partial z}\right], \tag{10.3}$$

are also known. The method of solution outlined in the paper may be used to solve other boundary problems of fundamental interest. For example, the skew-symmetric problem of specifying displacements u, v on the surfaces of the geometric discontinuity may be formulated and solved in a similar manner.

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Three-dimensional elastostatic problems for an infinite solid with geometric discontinuities are formulated and solved with the aid of potential functions. In the problem of a linearly varying pressure specified over a plane region bounded by an ellipse, use is made of the gravitational potential at an exterior point of a homogeneous elliptical disk. The problem of prescribing displacements on the surfaces of discontinuity is governed by the Newtonian potential of a simple layer distributed over a disk in the shape of the region of discontinuity. The mass density of the disk is proportional to the prescribed normal displacement. For an "elliptically shaped" region, the application of the symmetrical form of ellipsoidal coordinate leads to an integral equation of the Abel type for the potential function. It is shown that if the displacements normal to the elliptical plane are given by a polynomial in the variables x^2 and y^2, then the corresponding normal stress acting over the ellipse is also a polynomial. The results given in this report are useful in the prediction of the stability behavior of elastic solids containing geometric discontinuities.

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